/Quadratics/

The discriminant $\mathbf{b^2 - 4ac}$

- If $\mathbf{b^2 - 4ac}$ is a perfect square, the roots will be integers or fractions
- If $\mathbf{b^2 - 4ac > 0}$, the equation $ax^2 + bx + c = 0$ will have two roots
- If $\mathbf{b^2 - 4ac < 0}$, there will be no roots
- If $\mathbf{b^2 - 4ac = 0}$, the roots given by $x = \frac{-b \pm 0}{2a} = \frac{-b}{2a}$, and there is one root only or repeated root

Example1: Use the value of the discriminant $b^2 - 4ac$ to determine whether the following equations have two roots, one root or no roots

- $\mathbf{2x^2 - 3x - 4 = 0}$

a=2, b=-3, c=-4, $b^2 - 4ac = (-3)^2 - 4 \cdot 2 \cdot (-4) = 9 + 32 = 41 > 0$

so the equation has two roots. Also, as 41 is not perfect square, the roots are irrational

- $\mathbf{2x^2 - 3x - 5 = 0}$

a=2, b=-3, c=-5, $b^2 - 4ac = (-3)^2 - 4 \cdot 2 \cdot (-5) = 9 + 40 = 49 > 0$

so the equation has two roots. Also, as 49 is perfect square, the roots are rational

- $\mathbf{2x^2 - 4x + 5 = 0}$

a=2, b=-4, c=5, $b^2 - 4ac = (-4)^2 - 4 \cdot 2 \cdot 5 = -24 < 0$

As the discriminant is negative, the equation $2x^2 - 4x + 5 = 0$

has no roots.

- $\mathbf{2x^2 - 4x + 2 = 0}$

a=2, b=-4, c=2, $b^2 - 4ac = (-4)^2 - 4 \cdot 2 \cdot 2 = 0$

As the discriminant is zero, the equation $2x^2 - 4x + 2 = 0$

has only one (repeated) root

Example2: **Find the value of k for which $kx^2 - 4x + 1 = 0$ has equal roots**

- $b^2 - 4ac = 0$

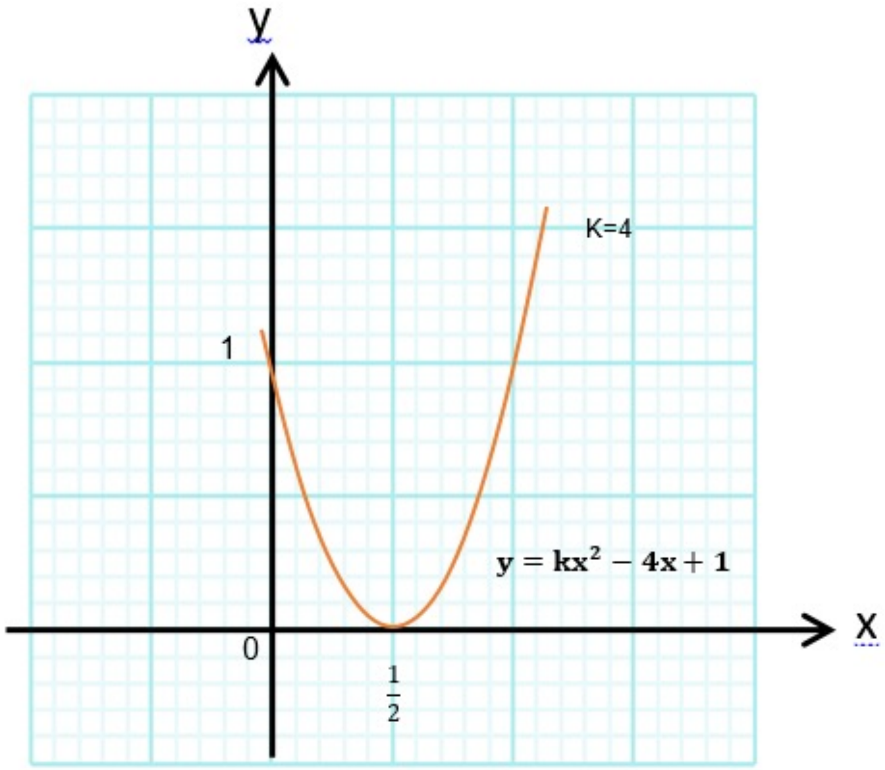
$$(-4)^2 - 4 \cdot k \cdot 1 = 0$$

$$4k = 16$$

$$k = 4$$

When $k = 4$

$$x = \frac{4 \pm \sqrt{0}}{2 \cdot 4} \Longrightarrow x = \frac{1}{2}$$



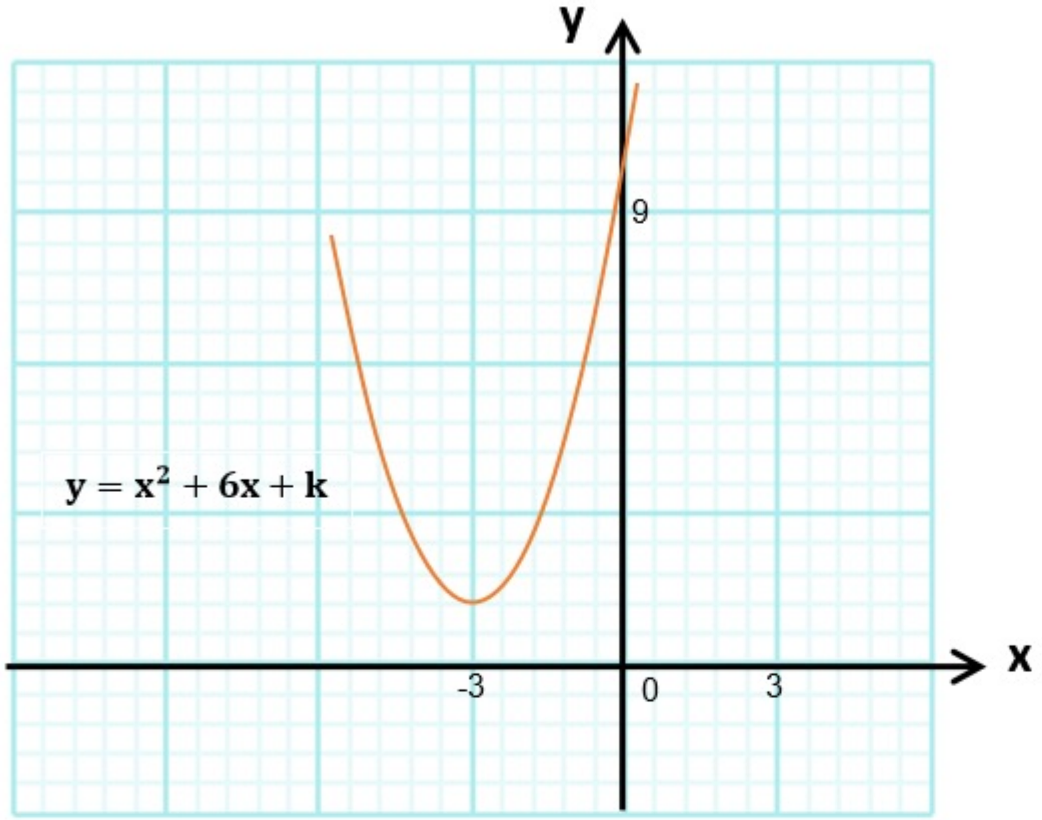
Example3: **Find the value of k for which $x^2 + 6x + k = 0$ has no real roots**

- $b^2 - 4ac < 0$

$$6^2 - 4 \cdot 1 \cdot k < 0$$

$$36 < 4k$$

$$k > 9$$



Example4: **Find the value of k for which $x^2 + kx + 9 = 0$ has different roots**

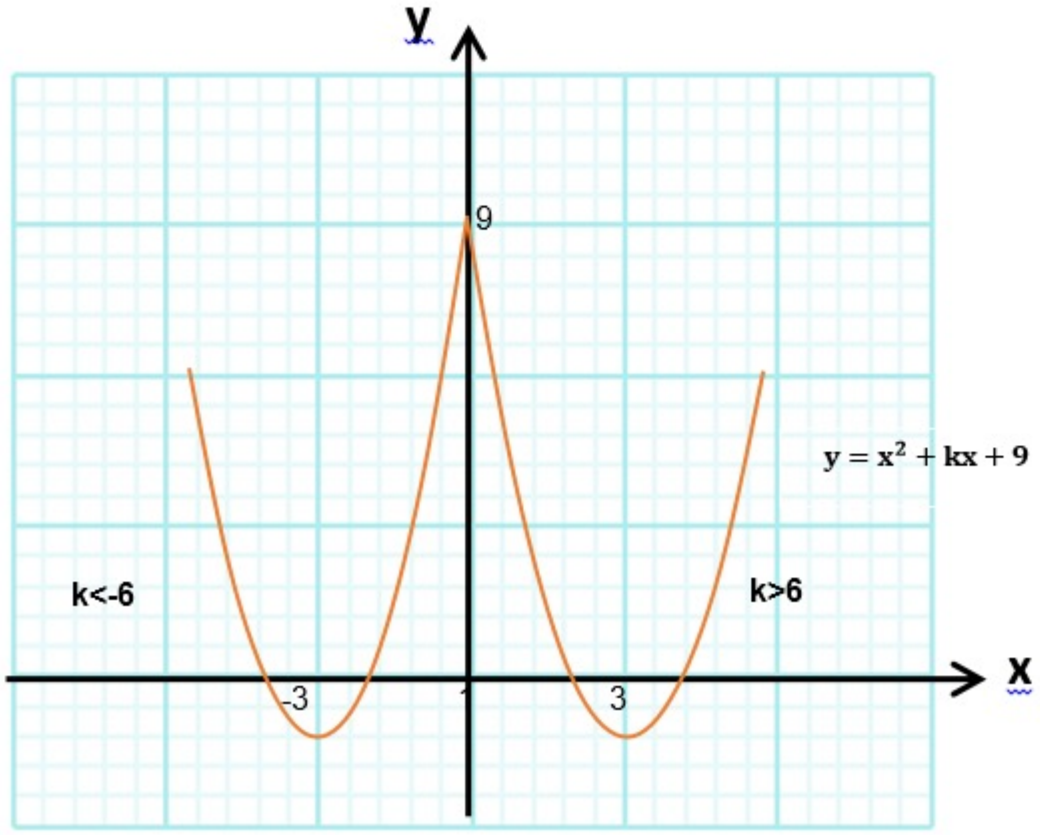
- $b^2 - 4ac > 0$

$$k^2 - 4 \cdot 1 \cdot 9 > 0$$

$$k^2 - 36 > 0$$

$$(k-6)(k+6) > 0$$

$$k < -6 \text{ and } k > 6$$



- Example5: **Find the points of intersection of the curve xy=6 and the line y=9-3x**

$x(9-3x)=6 \ \rightarrow \ 3x^2 - 9x + 6 = 0$

$\boldsymbol{b^2 - 4ac} = (-9)^2 - 4\cdot 3\cdot 6 = 9 > 0$

The equation has two solution, so the line meets the curve at two points

$$x_{1,2} = \frac{9 \pm \sqrt{9}}{2\cdot 3} = \frac{9 \pm 3}{6} \rightarrow x_1 = 1, x_2 = 2$$

$y_1 = 6\ , y_2 = 3 \quad \rightarrow \quad (1{,}6), (2{,}3)$

- Example6: Find the value of k for which $kx^2 - 2x - 7 = 0$ has two real roots

$$(-2)^2 - 4 \cdot k \cdot (-7) = 4 + 28k > 0$$

$$k > -\frac{1}{7}$$

- Example7: Find the value of k for which $kx^2 - 2x - 7 = 0$ has repeated roots

$$2^2 - 4 \cdot 3 \cdot k = 4 - 12k = 0 \quad k = \frac{1}{3}$$

- Example8 : Find the points of intersection of the curve $x^2 + y^2 = 25$ and the line y+2x=5

$$x^2 + (5 - 2x)^2 = 25 \rightarrow 5x^2 - 20x = 0$$

$$5x(x - 4) = 0 \rightarrow x_1 = 0, x_2 = 4 \quad y_1 = 5, y_2 = -3$$

$$(0, 5), (4, -3)$$

The line $y = \frac{x}{k} + k$, where k is a constant, is a tangent to the curve $4y = x^2$ at the point P. Find

(i) the value of k, [3]

(ii) the coordinates of P. [3]

- **(i)** $\frac{x^2}{4} = \frac{x}{k} + k \rightarrow kx^2 - 4x - 4k^2 = 0$

Tangent = $\mathbf{b^2 - 4ac = 0}$

$$(-4)^2 - 4 \cdot k \cdot (-4k^2) = 0$$
$$16 + 16k^3 = 0 \rightarrow k = -1$$

- **(ii)** $y = -x - 1\,,\quad 4y = x^2 \rightarrow\; 4(-x-1) = x^2$

$x^2 + 4x + 4 = 0 \quad \rightarrow \quad x = -2, \qquad y = 1$

P(-2,1)

A straight line has equation $y = -2x + k$, where k is a constant, and a curve has equation $y = \frac{2}{x-3}$.

(i) Show that the x-coordinates of any points of intersection of the line and curve are given by the equation $2x^2 - (6+k)x + (2+3k) = 0$. [1]

(ii) Find the two values of k for which the line is a tangent to the curve. [3]

The two tangents, given by the values of k found in part (ii), touch the curve at points A and B.

(iii) Find the coordinates of A and B and the equation of the line AB. [6]

- (i) $\frac{2}{x-3} = -2x + k \rightarrow -2x^2 + 6x + kx - 3k - 2 = 0$

  $-2x^2 + (6+k)x - (3k+2) = 0$

  $\mathbf{2x^2 - (6+k)x + (2+3k) = 0}$

- (ii) Tangent = $\mathbf{b^2 - 4ac = 0}$

  $(6+\text{k})^2 - 4 \cdot 2 \cdot (2+3k) = 0 \rightarrow \boldsymbol{k = 2 \ or \ 10}$

- (iii) $y = -2x + 2, \ y = \frac{2}{x-3} \rightarrow 2x^2 - 8x + 8 = 0 \rightarrow (x-2)^2 = 0$

$\boldsymbol{A(2, -2)}$

$y = -2x + 10, \qquad y = \frac{2}{x-3} \rightarrow 2x^2 - 16x + 32 = 0 \rightarrow (x-4)^2 = 0$

$\boldsymbol{B(4, 2)}$

$AB\text{:} \ -2 = 2m + c, \qquad 2 = 4m + c \rightarrow -2 - 2m = 2 - 4m$

$m = 2, c = -6 \rightarrow$ **Equation of the line AB is $y = 2x - 6$**

(i) Express $2x^2 - 10x + 8$ in the form $a(x+b)^2 + c$, where a, b and c are constants, and use your answer to state the minimum value of $2x^2 - 10x + 8$. [4]

(ii) Find the set of values of k for which the equation $2x^2 - 10x + 8 = kx$ has no real roots. [4]

- (i) $2x^2 - 10x + 8 = 2(x^2 - 5x) + 8 =$

$$= 2\left(x^2 - 2\cdot\frac{5}{2}x + \frac{25}{4} - \frac{25}{4}\right) + 8 = 2\left(x - \frac{5}{2}\right)^2 - \frac{25}{2} + 8 =$$

$= 2(x - 2\frac{1}{2})^2 - 4\frac{1}{2}$; **min value is $-4\frac{1}{2}$ allow $(2\frac{1}{2}, -4\frac{1}{2})$**

- **(ii)** $2x^2 - 10x + 8 - kx = 0 \rightarrow 2x^2 - (10+k)x + 8 = 0$

$b^2 - 4ac < 0$

$(10+k)^2 - 64 < 0 \rightarrow k^2 + 20k + 36 < 0$

$k = -18$ or -2 **$-18 < k < -2$**